International Journal of Theoretical Physics, Vol. 25, No. 12, 1986

Separability in Distant Jauch-Type Hybrid Macrostates of a Quantum and a Classical System

Fedor Herbut[1]

Received March 11, 1986

It is assumed that for a quantum system (Q) plus a classical one (C) that are in a distant state the actually measurable Hermitian operators are of the form $A\otimes\sum_{k\in K} b_k Q_k$ (A is any Hermitian operator for Q, and the decomposition $\sum_k Q_k = 1$ of the identity is, after von Neumann, characteristic for C). This leads to Jauch-type macrostates (classes of microstates or statistical operators) for $Q+C$. On the other hand, it is shown that in the $Q+Q$ case the essence of quantum correlations are the conditional states (or statistical operators) of subsystem I and the reduced state ρ_{II}. Along these lines, the correlation entities (as a complete set of invariants) for the macrostates of the $Q+C$ system are derived, and it is shown that one can make an isomorphic transition from the σ-convex set of the latter to that of the hybrid macrostates (ρ_k, p_k). Here ρ_k is the conditional state of Q under the condition that Q_k occurs on C, and p_k is a classical discrete probability distribution on K, taking the place of ρ_{II} as the macrostate of C. This study indirectly throws new light on the nonseparability in the $Q+Q$ case by contrasting it with a well-understood separability in the $C+C$ and $Q+C$ cases.

1. INTRODUCTION

Jauch (1964, 1968) defined a classical system as a quantum object on which one can measure only an Abelian set $\mathbf{O}^c$ of Hermitian operators. Further, he showed that any restricted set $\mathbf{O}'$ of observables induces an equivalence relation $\sim$ in the set $\mathbf{S}$ of all microstates (statistical operators) of a quantum system through the definition

$$\rho\sim\rho',\qquad \rho,\rho'\in\mathbf{S},\qquad \text{if}\quad \forall A\in\mathbf{O}':\quad \operatorname{Tr} A\rho=\operatorname{Tr} A\rho' \tag{1}$$

The physical meaning of $\sim$ is "indistinguishable by measurement of any observable from $\mathbf{O}'$."

Jauch proved the following theorem.

[1]Department of Physics, Faculty of Science, 11001 Belgrade, Yugoslavia.

0020-7748/86/1200-1215$05.00/0

Theorem on Induced σ-Convexity. The set of classes $\mathbf{S}/\sim$ is σ-convex due to the σ-convexity of $\mathbf{S}$ via arbitrary class representatives: if $C_1, C_2, \ldots \in \mathbf{S}/\sim$ (a finite or countably infinite set of classes), and $w_1 > 0, w_2 > 0, \ldots, \sum_i w_i = 1$, then also $\sum_i w_i C_i \in \mathbf{S}/\sim$. To evaluate $\sum_i w_i C_i$, one takes arbitrarily $\rho_1 \in C_1, \rho_2 \in C_2, \ldots$, and $\sum_i w_i C_i$ is the equivalence class to which $\sum_i w_i \rho_i$ belongs.

Jauch called the equivalence classes (elements of $\mathbf{S}/\sim$) for a classical system *macrostates.*

In a recent investigation of Jauch's approach to the quantum theory of measurement (Herbut, 1986), Jauch's set $\mathbf{O}^c$ for a classical system was given the more specific form of von Neumann (von Neumann, 1955, Chapter V, Section 4)

$$\mathbf{O}^c(B_0) \equiv \left\{ \sum_{k\in K} b_k Q_k\text{: Herm. op., } \sum_{k\in K} Q_k = 1 \text{ fixed} \right\} \tag{2}$$

where Q_k are the eigenprojectors of the *basic observable*

$$B_0 = \sum_{k\in K} b_k^0 Q_k \qquad (k \neq k' \Rightarrow b_k^0 \neq b_{k'}^0)$$

The set $\mathbf{O}^c(B_0)$ consists of all functions of B_0 that are Hermitian operators.

In an attempt to make Jauch's approach less unrealistic, states of the quantum system (Q) and the classical object (C) were envisaged (Herbut, 1986) that were *distant,* i.e., in which only coincidence measurements were performable. More fully stated, only Hermitian operators of the form $A \otimes B$, A any observable for the quantum system, and $B \in \mathbf{O}^c(B_0)$, were assumed to be measurable on $Q + C$. This set of observables was denoted by $\mathbf{O} \otimes \mathbf{O}^c(B_0)$. Applying Jauch's equivalence relation (1) with $\mathbf{O}' \equiv \mathbf{O} \otimes \mathbf{O}^c(B_0)$ to the microstates of $Q + C$, the macrostates were determined. This, of course, extended the nonmeasurability of those Hermitian operators that were outside $\mathbf{O} \otimes \mathbf{O}^c(B_0)$ from the distant to all states of $Q + C$. We have a clear physical understanding of this assumption only for states in which Q and C are sufficiently spatially separated, though they may have interacted in the past and now contain distant correlations as a consequence [see the last section in Vujičić and Herbut (1984)]. It may be advisable to restrict the use of the Jauch-type approach at issue to distant $Q + C$ states. [In Herbut (1986) the distant microstates were the states of $Q + C$ after the measurement interaction has ceased. Jauch's explanation of collapse, i.e., its disappearance in terms of the macrostates, was found to hold formally for most general measurements.]

To introduce the subject of the investigation in this paper, the correlations are given a definition and are analysed for the $Q + Q$ case in the next section because in the approach adopted, the $Q + Q$ case is more basic than the $Q + C$ one.

In Section 3 an initial discussion of distant correlations in the $Q+C$ case is presented. It gives rise to a relevant mathematical problem. This is solved in Section 4. In Section 5 the main result, expressing the macrostates of $Q+C$ in terms of the basic distant correlation entities, is derived. In Section 6 separability in the cases $C+C$ and $Q+C$ is discussed.

2. CORRELATIONS IN THE $Q+Q$ CASE

2.1. Coincidences

We assume that we have two quantum systems I and II. Let ρ be their general state (statistical operator). Let P be an arbitrary event (a projector) for I and Q an arbitrary event (a projector) for II. The measurement of $P\otimes Q$ is called a coincidence measurement. The probability of its result 1 is

$$\mathrm{Tr}_{\mathrm{I,II}}(P\otimes Q)\rho \tag{3}$$

Since $(P\otimes 1)$ and $(1\otimes Q)$ commute, we can apply to (3) the usual factorization into the absolute probability of the occurrence of $(1\otimes Q)$ in ρ, and the conditional probability of the happening of P under the condition that $(1\otimes Q)$ took place in ρ:

$$\mathrm{Tr}_{\mathrm{I,II}}(P\otimes Q)\rho = (\mathrm{Tr}_{\mathrm{II}} Q\rho_{\mathrm{II}})\cdot \mathrm{Tr}_{\mathrm{I}} P[(\mathrm{Tr}_{\mathrm{II}} Q\rho_{\mathrm{II}})^{-1}\,\mathrm{Tr}_{\mathrm{II}}(1\otimes Q)\rho] \tag{4}$$

Here $\rho_{\mathrm{II}} \equiv \mathrm{Tr}_{\mathrm{I}}\,\rho$ is the reduced state (statistical operator) for II, and it is assumed that $\mathrm{Tr}_{\mathrm{II}} Q\rho_{\mathrm{II}} > 0$.

The operator $\mathrm{Tr}_{\mathrm{II}}(1\otimes Q)\rho$ is a positive operator in the state space $\mathscr{H}_{\mathrm{I}}$, as immediately seen from

$$\forall|\psi\rangle\in\mathscr{H}_{\mathrm{I}}:\quad \langle\psi|\mathrm{Tr}_{\mathrm{II}}(1\otimes Q)\rho|\psi\rangle = \mathrm{Tr}_{\mathrm{I,II}}(|\psi\rangle\langle\psi|\otimes Q)\rho \geq 0$$

[a special case of (3)]. Further,

$$\rho_{\mathrm{I}}(Q) \equiv (\mathrm{Tr}_{\mathrm{II}}\, Q\rho_{\mathrm{II}})^{-1}\,\mathrm{Tr}_{\mathrm{II}}(1\otimes Q)\rho \tag{5}$$

[the entity in the square brackets in (4)] obviously implies $\mathrm{Tr}_{\mathrm{I}}\rho_{\mathrm{I}}(Q)=1$. Hence, (5) defines a statistical operator for I. We call it *the conditional state* under the condition that $(1\otimes Q)$ occurred in ρ. Note that $\rho_{\mathrm{I}}(1)=\rho_{\mathrm{I}}\equiv \mathrm{Tr}_{\mathrm{II}}\,\rho$ is the reduced state for I.

The conditional state $\rho_{\mathrm{I}}(Q)$ was utilized in previous work (Herbut and Vujičić, 1976, Section 6.B). It is also recognizable in the work of Ozawa (1984, Lemma 2.1).

It is an important fact that every two distinct states of I+II can be told apart by a coincidence measurement. [Historically, this goes back to Furry (1936).] More precisely,

$$\forall\rho\neq\rho',\quad \exists(P\otimes Q):\quad \mathrm{Tr}_{\mathrm{I,II}}(P\otimes Q)\rho \neq \mathrm{Tr}_{\mathrm{I,II}}(P\otimes Q)\rho' \tag{6}$$

[It is straightforward, but somewhat lengthy, to prove (6) *ab contrario*, starting with a basis in $\mathscr{H}_{\rm I}$ and one in $\mathscr{H}_{\rm II}$.] The following result is an immediate consequence of (6) and (4).

Corollary 1. If $\rho \neq \rho'$ but $\rho_{\rm II} \equiv {\rm Tr}_{\rm I}\,\rho = \rho'_{\rm II} \equiv {\rm Tr}_{\rm I}\,\rho'$, then there exists an event Q for II such that

$$\rho_{\rm I}(Q) \neq \rho'_{\rm I}(Q) \tag{7}$$

where the lhs and the rhs are defined through (5) by ρ and ρ', respectively.

Corollary 1 implies in turn:

Corollary 2. If $\rho \neq \rho'$ but $\rho_i = \rho'_i$, $i = {\rm I}, {\rm II}$ (the reduced statistical operators), then $\exists Q$, a projector for II, such that

$$\rho_{\rm I}(Q) \neq \rho'_{\rm I}(Q)$$

Clearly, in view of (4) and Corollaries 1 and 2, the conditional states implied by an arbitrary composite state ρ can be considered, mathematically, to be the essence of the correlations in it. But physically this has meaning only when the coincidence measurements $P \otimes Q$ can actually be performed. If I and II are spatially close to each other, the subsystem measurements of $P \otimes 1$ and $1 \otimes Q$, making up, with the help of some coincidence arrangement, the coincidence, cannot be carried out because the measuring instrument is bound to affect also the other subsystem. Therefore, the physical meaning of (4) for nondistant states ρ is dubious, and it is wise to restrict oneself to distant states. Hence, we say that the conditional states $\rho_{\rm I}(Q)$ are *the essence of the distant correlations* in a distant state ρ.

Subsystems I and II play symmetrical roles in the $Q+Q$ case at issue. Hence, the relation symmetrical to (5) gives $\rho_{\rm II}(P)$, the conditional states for subsystem II.

2.2. The Special Case of Pure Distant Composite States

If ρ is a *pure* distant state $|\phi\rangle\langle\phi|$, and one takes for Q any elementary event (atom) $|\varphi\rangle\langle\varphi|$, then (5) gives, as is is easily seeen, a pure state

$$\rho_{\rm I}(Q) = |\psi\rangle\langle\psi|$$

and (5) itself can be replaced by

$$|\varphi\rangle = \langle\varphi|\phi\rangle\!\rangle \|\langle\varphi|\phi\rangle\!\rangle\|^{-1}$$

Here $\langle\cdots|\cdots\rangle\!\rangle$ stands for the partial scalar product over $\mathscr{H}_{\rm II}$ (the counterpart of ${\rm Tr}_{\rm II}$), and $\|\cdots\|$ denotes the norm (cf. Herbut and Vujičić, 1976, Appendix 1).

Since $\langle\varphi|\phi\rangle\!\rangle = A_a|\varphi\rangle$ defines an antilinear mapping A_a of $\mathscr{H}_{\rm II}$ into $\mathscr{H}_{\rm I}$, it is suitable to replace $|\phi\rangle$ by A_a in studying the distant correlations inherent in $|\phi\rangle$ (Herbut and Vujičić, 1976; Vujičić and Herbut, 1984).

It has been shown (Herbut and Vujičić, 1976; Vujičić and Herbut, 1984) that not only is the above state $|\psi\rangle\langle\psi|$ the conditional state of I under the condition of occurrence of $|\varphi\rangle\langle\varphi|$ in the state $|\phi\rangle$, but also vice versa, $|\varphi\rangle\langle\varphi|$ is the conditional state of II under the condition $|\psi\rangle\langle\psi|$ in $|\phi\rangle$ if and only if

$$[|\varphi\rangle\langle\varphi|, \rho_{II}] = 0$$

or, equivalently, $[|\psi\rangle\langle\psi|, \rho_I] = 0$. In this case $|\psi\rangle\langle\psi|$ and $|\varphi\rangle\langle\varphi|$ are called twins. They are related by $|\psi\rangle = U_a|\varphi\rangle$, where U_a, the correlation operator implied by $|\phi\rangle$, is obtained as the antiunitary polar factor of A_a:

$$A_a = U_a \rho_{II}^{1/2}, \qquad \rho_{II} \equiv \mathrm{Tr}_I |\phi\rangle\langle\phi|$$

3. CORRELATIONS IN A DISTANT MACROSTATE OF $Q+C$

Returning to the states ρ of the $Q+C$ case, we investigate the subsystem states ρ_{II} and the conditional states $\rho_I(Q)$ of ρ.

Theorem 1. Let **S** be the σ-convex set of all microstates (statistical operators) of $Q+C$, and let $\sim$ in **S** be defined by (1) with $\mathbf{O}' \equiv \mathbf{O} \otimes \mathbf{O}^c(B_0)$ [cf. (2)].

(A) Any two equivalent microstates ρ and ρ' imply through (5) for any event (projector) $Q \in \mathbf{O}^c(B_0)$ *the same* conditional state $\rho_I(Q)$. In particular, the reduced states for I coincide.

(B) If $\rho \sim \rho'$, $\rho, \rho' \in \mathbf{S}$, and $\rho_{II} \equiv \mathrm{Tr}_I \rho$, $\rho'_{II} \equiv \mathrm{Tr}_I \rho'$, then ρ_{II} and ρ'_{II} are *equivalent* via (1) with $\mathbf{O}' \equiv \mathbf{O}^c(B_0)$.

Proof. (A) Let $|\psi\rangle \in \mathcal{H}_I$. Then it follows from (5) that

$$\langle\psi|\rho_I(Q)|\psi\rangle = (\mathrm{Tr}_{II}\, Q\rho_{II})^{-1}\, \mathrm{Tr}_{I,II}(|\psi\rangle\langle\psi| \otimes Q)\rho \tag{8a}$$

$$\langle\psi|\rho'_I(Q)|\psi\rangle = (\mathrm{Tr}_{II}\, Q\rho'_{II})^{-1}\, \mathrm{Tr}_{I,II}(|\psi\rangle\langle\psi| \otimes Q)\rho' \tag{8b}$$

Since $\mathrm{Tr}_{II}\, Q\rho_{II} = \mathrm{Tr}_{I,II}(1 \otimes Q)\rho$, $\mathrm{Tr}_{II}\, Q\rho'_{II} = \mathrm{Tr}_{I,II}(1 \otimes Q)\rho'$, and $(1 \otimes Q), (|\psi\rangle\Delta \langle\psi| \otimes Q) \in \mathbf{O} \otimes \mathbf{O}^c(B_0)$, the rhs of (8a) and (8b) are equal. The arbitrariness of $|\psi\rangle$ then gives $\rho_I(Q) = \rho'_I(Q)$ as claimed.

(B) Let $B \in \mathbf{O}^c(B_0)$. Then $(1 \otimes B) \in \mathbf{O} \otimes \mathbf{O}^c(B_0)$, and

$$\mathrm{Tr}_{II}\, B\rho_{II} \equiv \mathrm{Tr}_{I,II}(1 \otimes B)\rho = \mathrm{Tr}_{I,II}(1 \otimes B)\rho' \equiv \mathrm{Tr}_{II}\, B\rho'_{II} \quad \blacksquare$$

Let $\mathbf{S}_{II}$ be the σ-convex set of all microstates in $\mathcal{H}_{II}$, and $\sim$ the equivalence relation in it defined by (1) with $\mathbf{O}' \equiv \mathbf{O}^c(B_0)$. As seen from Theorem 1(B), elements from $\mathbf{S}_{II}/\sim$ are subsystem states of C in a distant macrostate of $Q+C$.

One wonders what the σ-convex structure of $\mathbf{S}_{II}/\sim$ is. Would it be possible to make an isomorphic transition from $\mathbf{S}_{II}/\sim$ into some σ-convex set that we know well, and the elements of which are single elements and not classes of entities? Since $\{Q_k: k \in K\}$ are practically elementary events (atoms) for C, the σ-convex set $\mathrm{s} \equiv \{p_k: p_k \geq 0,\ k \in K;\ \sum_{k' \in K} p_{k'} = 1\}$ of discrete probability distributions on K with

$$\forall k \in K: \quad p_k \equiv \mathrm{Tr}_{\mathrm{I,II}}(1 \otimes Q_k)\rho = \mathrm{Tr}_{\mathrm{II}} Q_k \rho_{\mathrm{II}}$$

appears to be relevant and promising in this respect. The σ-convex set s is what is called the classical discrete (CD) case.

This question is settled in the next section.

4. THE ROLE OF THE CD CASE

In view of definition (2) of $\mathbf{O}^c(B_0)$, one can obviously replace the latter in the equivalence relation (1) in the set of microstates S of a classical object by $\{Q_k: k \in K\}$.

Theorem 2. Let S be the σ-convex set of all statistical operators ρ of a classical object, and let

$$\rho \sim \rho' \qquad \text{if} \quad \forall k \in K: \qquad \mathrm{Tr}\, \rho Q_k = \mathrm{Tr}\, \rho' Q_k \tag{9}$$

define an equivalence relation in S. The quotient set $\mathrm{S}/\sim$ is a σ-convex set (according to the theorem on induced convexity). Let, on the other hand, s be the σ-convex set of all probability distributions $\{p_k: k \in K\}$ ($\forall k \in K: p_k \geq 0,\ \sum_{k \in K} p_k = 1$) on K. The map $f': \mathrm{S} \to \mathrm{s}$ giving $p_k \equiv f'(\rho)$ that is determined by

$$\forall \rho \in \mathrm{S}, \quad \forall k \in K: \qquad p_k \equiv \mathrm{Tr}\, \rho Q_k \tag{10}$$

induces an *isomorphism* f of the σ-convex set $\mathrm{S}/\sim$ onto the σ-convex set s.

Proof. Clearly, $\sum_{k \in K} p_k = \mathrm{Tr}\, \rho(\sum_{k \in K} Q_k) = \mathrm{Tr}\, \rho = 1$, hence f' is a map of S into s. Let $p_k \in \mathrm{s}$ be arbitrary. We take $\forall k \in K: |\varphi_k\rangle \in R(Q_k)$ (the range of Q_k), $\langle \varphi_k | \varphi_k \rangle = 1$, and we construct $\rho \equiv \sum_{k \in K} p_k |\varphi_k\rangle\langle\varphi_k| \in \mathrm{S}$. Then f' maps ρ back into p_k. Hence, f' is a surjection. Definition (9) obviously can be construed as saying that $\rho \sim \rho'$ if and only if $f'(\rho) = f'(\rho')$. Thus, f' induces a bijection f of $\mathrm{S}/\sim$ onto s.

Further, let $\rho = \sum_i w_i \rho_i$, $\rho \in \mathrm{S}$, $\forall i: \rho_i \in \mathrm{S}$, $w_i > 0$, $\sum_i w_i = 1$. The application of f' gives $\forall k \in K: p_k^{(i)} \equiv \mathrm{Tr}\, \rho_i Q_k$. Then, denoting $f'(\rho)$ by p_k, we have $\forall k \in K: p_k \equiv \mathrm{Tr}\, \rho Q_k = \sum_i w_i p_k^{(i)}$. Consequently, f' is a homomorphism of the σ-convex set S onto the σ-convex set s, and f is a homomorphic bijection of $\mathrm{S}/\sim$ onto s.

Finally, for f to be an isomorphism, as claimed, also the inverse map f^{-1} taking s onto $\mathrm{S}/\sim$ must preserve the σ-convex combination. This is always true for a homomorphic bijection in a set closed under an operation:

if $f'(\rho)=\sum_i w_i f'(\rho_i)$, $\forall i$: $w_i>0, \sum_i w_i=1$, then it cannot happen that $\rho \neq \sum_i w_i\rho_i$, because $f'(\sum_i w_i\rho_i)=\sum_i w_i f'(\rho_1)$. ■

We call p_k, the elements of **s**, the *CD macrostates.* Evidently, if two microstates ρ and ρ' of a classical object are equivalent [see (9)], then the corresponding average values of any $B\in \mathbf{O}^c(B_0)$ are equal. Thus, one can speak of the average of B in a macrostate (element of $\mathbf{S}/\sim$).

When the isomorphic transition f takes the quantum macrostates into the CD ones, the observables B become the CD variables b_k [cf. (2)].

Corollary 3. The isomorphism f of Theorem 2 does not change the expectation value of any measurable Hermitian operator $B\in \mathbf{O}^c(B_0)$:

$$\forall \rho \in \mathbf{S}: \qquad \operatorname{Tr} \rho B = \sum_{k\in K} p_k b_k$$

where p_k is given by $f'(\rho)$.

Proof. The spectral form of B [cf. (2)] and (10) give immediately $\operatorname{Tr} \rho B=\sum_{k\in K} p_k b_k$ as claimed. ■

Let us return to the results of Theorem 1 concerning the correlation entities in a macrostate ρ of $Q+C$. Theorem 2 enables us to replace the class of equivalent reduced statistical operators ρ_{II} of ρ by the CD macrostates p_k:

$$\forall k\in K: \qquad p_k \equiv \operatorname{Tr}_{\text{II}} Q_k \rho_{\text{II}} \equiv \operatorname{Tr}_{\text{I,II}}(1\otimes Q_k)\rho \tag{11}$$

Further, in the set $\{\rho_{\text{I}}(Q)\colon Q\in \mathscr{P}(\mathscr{H}_{\text{II}})\cap \mathbf{O}^c(B_0)\}$ of all conditional states for the quantum subsystem [$\mathscr{P}(\mathscr{H}_{\text{II}})$ being the set of all projectors in $\mathscr{H}_{\text{II}}$] there is some redundancy, as the following result shows.

Theorem 3. Let $Q\in \mathscr{P}(\mathscr{H}_{\text{II}})$, and $Q\in \mathbf{O}^c(B_0)$. Then one can write $Q=\sum_{k\in K} b_k Q_k$, with

$$\forall k\in K: \qquad b_k \equiv \delta(k\in K_Q) \equiv \begin{cases} 1 & \textit{if } k\in K_Q \\ 0 & \text{otherwise} \end{cases}$$

where $K_Q\equiv \{k\colon QQ_k=Q_k\}$. If ρ is a microstate of $Q+C$, and it implies, via (11), p_k as its CD macrostate for subsystem II, then, defining $p(K_Q)\equiv \sum_{k\in K_Q} p_k$, if $p(K_Q)>0$, we have

$$\rho_{\text{I}}(Q) = \sum_{k\in (K_Q\cap \bar{K})} \frac{p_k}{p(K_Q)} \rho_k \tag{12}$$

where $\bar{K}\equiv \{k\colon p_k>0, k\in K\}$ is the support of p_k, and

$$\forall k\in K: \qquad \rho_k \equiv \rho_{\text{I}}(Q_k) = p_k^{-1} \operatorname{Tr}_{\text{II}}(1\otimes Q_k)\rho \tag{13}$$

We call $\{\rho_k: k \in \bar{K}\}$ the set of *basic conditional states* of the quantum subsystem in a macrostate of a $Q+C$ system.

Proof of Theorem 3. The claims about the CD variable b_k corresponding to Q are obvious from (2). To derive (12), we insert $Q = \sum_{k \in K} \delta(k \in K_Q) Q_k$ on the rhs of (5):

$$\rho_{\mathrm{I}}(Q) = p(K_Q)^{-1} \sum_{k \in K_Q} \mathrm{Tr}_{\mathrm{II}}(1 \otimes Q_k)\rho$$

As seen from (11), for $p_k = 0$, $k \in K_Q$, $\mathrm{Tr}_{\mathrm{II}}(1 \otimes Q_k)\rho = 0$ (because it is a positive operator). For $p_k > 0$, we insert the factor p_k/p_k to obtain (12). ■

There are some interesting consequences of Theorems 2 and 3.

Corollary 4. The expectation value of any observable $(A \otimes \sum_{k \in K} b_k Q_k) \in \mathbf{O} \otimes \mathbf{O}^c(B_0)$ in any microstate ρ of $Q+C$ depends on no other constituent of ρ but the invariant entities p_k, $\{\rho_k: k \in \bar{K}\}$ determined by ρ:

$$\left\langle A \otimes \sum_{k \in K} b_k Q_k, \rho \right\rangle = \sum_{k \in K} b_k p_k \, \mathrm{Tr}_{\mathrm{I}} \, A \rho_k \tag{14}$$

(if $p_k = 0$, ρ_k may be undefined).

Proof is obtained by immediate evaluation.

Corollary 5. Let $\mathbf{S}$ be the σ-convex set of all $Q+C$ microstates, and let $\sim$ be defined through (1) with $\mathbf{O}' \equiv \mathbf{O} \otimes \mathbf{O}^c(B_0)$. If $\rho \not\sim \rho'$, and p_k, $\{\rho_k: k \in \bar{K}\}$ are the invariant entities corresponding to ρ, and p'_k, $\{\rho'_k: k \in \bar{K}'\}$ those corresponding to ρ', then either $\exists k \in K: p_k \neq p'_k$ and/or $\exists k: k \in \bar{K} \cap \bar{K}'$, $\rho_k \neq \rho'_k$.

Proof. Since $\rho \not\sim \rho'$ means that $\exists (A \otimes \sum_{k \in K} b_k Q_k) \in \mathbf{O} \otimes \mathbf{O}^c(B_0)$ such that the lhs of (14) for ρ and for ρ' are distinct, the same is true for the rhs. This cannot be so unless the claim of Corollary 5 is true. ■

Thus, considering the classes in $\mathbf{S}$ and their invariant entities p_k, $\{\rho_k: k \in \bar{K}\}$, the latter reflect not only the "sameness" within the former, but also their distinctness. This brings us to the conjecture that a macrostate of $Q+C$ should be expressible in terms of the invariant entities.

In the next section we prove this conjecture.

5. MACROSTATES OF $Q+C$ IN TERMS OF THE INVARIANT ENTITIES

To begin with, we define a new set $\mathcal{M}_Q$: Let p_k be an arbitrary CD probability distribution on K, $\bar{K}$ its support, and $\{\rho_k: k \in \bar{K}\}$ arbitrary statistical operators in $\mathcal{H}_\mathrm{I}$. Defining $\rho_k \equiv 0$ for those k values for which $p_k = 0$, we write this as $M \equiv \{(\rho_k, p_k): k \in K\}$ or briefly as $M \equiv (\rho_k, p_k)$. The set $\mathcal{M}_Q$ is the set of all such M.

Now we introduce the operation of (finite or countably infinite) convex combinations in $\mathcal{M}_Q$, i.e., we make the latter σ-convex.

Let $w_q > 0$, $q = 1, 2, \ldots, \sum_q w_q = 1$, be an arbitrary sequence of statistical weights, and let $\{M_q: q = 1, 2, \ldots\}$ be an arbitrary sequence of elements of $\mathcal{M}_Q$. We define $\sum_q w_q M_q$ as follows: If $M_q = (\rho_k^{(q)}, p_k^{(q)})$ and $M \equiv \sum_q w_q M_q = (\rho_k, p_k)$, then

$$\forall k \in K: \qquad p_k \equiv \sum_q w_q p_k^{(q)} \tag{15a}$$

$$\rho_k \equiv \sum_q w_q^{(k)} \rho_k^{(q)} \tag{15b}$$

where the weights $w_q^{(k)}$ are determined by

$$\forall q: \qquad w_q^{(k)} \equiv w_q p_k^{(q)} / p_k \tag{15c}$$

[they are statistical weights due to (15a)]. If $p_k = 0$, then $p_k^{(q)} = 0$, $q = 1, 2, \ldots$ [see (15a)], and (15c) is understood to give $w_q^{(k)} = 0$, $q = 1, 2, \ldots$ (though p_k^{-1} is undefined).

Next, we define the set $\mathbf{O} \times \mathbf{v}$ consisting of all pairs (A, b_k), where A is any Hermitian operator in $\mathcal{H}_\mathrm{I}$, and $\{b_k: k \in K\}$ are real numbers such that $\sum_{k \in K} b_k Q_k$ is a Hermitian operator in $\mathcal{H}_\mathrm{II}$.

Finally, we define the *expectation value* of any $(A, b_k) \in \mathbf{O} \times \mathbf{v}$ in any $M = (\rho_k, p_k) \in \mathcal{M}_Q$:

$$\langle (A, b_k), M \rangle \equiv \sum_{k \in K} b_k p_k \operatorname{Tr}_\mathrm{I} A \rho_k \tag{16}$$

It preserves the σ-convex combination. Namely, making use of (15b), (15c), (16) gives

$$\begin{aligned}
\langle (A, b_k), (\rho_k, p_k) \rangle &= \sum_q \sum_{k \in K} w_q^{(k)} b_k p_k \operatorname{Tr}_\mathrm{I} A \rho_k^{(q)} \\
&= \sum_q w_q \sum_{k \in K} b_k p_k^{(q)} \operatorname{Tr}_\mathrm{I} A \rho_k^{(q)} \\
&= \sum_q w_q \langle (A, b_k), (\rho_k^{(q)}, p_k^{(q)}) \rangle
\end{aligned}$$

Now we are prepared to resort to the main result of this investigation.

Theorem 4. Let $\mathbf{S}/\sim$ be the set of all macrostates of $Q+C$. For each of its elements C there exists one and only one $M \in \mathcal{M}_Q$ such that each $(A \otimes \sum_{k \in K} b_k Q_k) \in \mathbf{O} \otimes \mathbf{O}^c(B_0)$ has the same expectation value in C as in M; and vice versa, for each $M \in \mathcal{M}_Q$ there exists precisely one $C \in \mathbf{S}/\sim$ fulfilling this condition. This bijection of $\mathbf{S}/\sim$ onto $\mathcal{M}_Q$ is obtainable by taking the invariant entities of an arbitrary element $\rho \in C$, and by writing them as M. The bijection is an *isomorphism* of the σ-convex set $\mathbf{S}/\sim$ onto the σ-convex set $\mathcal{M}_Q$.

Proof. Let $C \in \mathbf{S}/\sim$, $\rho \in C$, and let $M \equiv \{\rho_k, p_k\} \in \mathcal{M}_Q$ be made up of the invariant entities of ρ. In view of their invariance within C (cf. Theorems 1 and 2), and due to Corollary 5, this transition from C to M is an injection of $\mathbf{S}/\sim$ into $\mathcal{M}_Q$. It is actually a bijection, as seen by taking an arbitrary $M \equiv (\rho_k, p_k) \in \mathcal{M}_Q$, by choosing for $\forall k \in K$: $|\varphi_k\rangle \in R(\rho_k)$, and by constructing

$$\rho \equiv \sum_{k \in \bar{K}} p_k \rho_k \otimes |\varphi_k\rangle\langle\varphi_k|$$

Evidently, $\rho \in \mathbf{S}$, and its invariant entities lead back to M.

The element $C \in \mathbf{S}/\sim$ and the corresponding $M \in \mathcal{M}_Q$ give the same expectation value for each observable from $\mathbf{O} \otimes \mathbf{O}^c(B_0)$, as is obvious from (14) and (16). For C there is no other $M' \in \mathcal{M}_Q$ with this property, or else C', corresponding to M', would give the same expectation value as C, and this would mean $C = C'$. Thus, the first two claims are established.

If $C = \sum_q w_q C_q$ and $\rho_1 \in C_1$, $\rho_2 \in C_2, \ldots$, $\rho \equiv \sum_q w_q \rho_k$, and we denote by $M_1, M_2, \ldots$ the corresponding elements of $\mathcal{M}_Q$, then, due to the fact that the expectation value preserves the σ-convex combination both on $\mathbf{S}/\sim$ and on $\mathcal{M}_Q$, $\sum_q w_q M_q$ gives the same expectation value for all elements of $\mathbf{O} \otimes \mathbf{O}^c(B_0)$ as C. Hence, as follows from the first two claims of Theorem 4, it is $\sum_q w_q M_q$ that corresponds to C. Thus, the correspondence is homomorphic. Since a homomorphic bijection of a σ-convex set onto another is an isomorphism (cf. the end of the proof of Theorem 2), Theorem 4 is proved. ■

6. SEPARABILITY IN THE $C+C$ AND $Q+C$ CASES

To understand the nature of the correlations in the $Q+C$ case, let us compare it with that in the $C+C$ case.

Let G and K be two countable sets of elementary events, and $p(g, k)$ an arbitrary probability distribution on $G \times K$. Let $p_k \equiv \sum_{g \in G} p(g, k)$ be its right marginal distribution, $\bar{K}$ the support of p_k, and $\forall k \in \bar{K}$: $p(g|k) \equiv p(g, k)/p_k$ the conditional (left) probabilities. Let us further gather into classes those k values that give equal conditional probabilities: $\bar{K}_n \equiv \{k: p_k > 0, p(g|k) \equiv p(g|n) \text{ indep. of } k\}$, $n = 1, 2, \ldots$, giving $\sum_n \bar{K}_n = \bar{K}$.

Finally, let

$$\forall n \qquad w_n \equiv \sum_{k \in \bar{K}_n} p_k = p(\bar{K}_n)$$

and

$$\forall n, \quad \forall k \in K: \qquad p(k|n) \equiv \delta(k \in \bar{K}_n) p_k / w_n.$$

[The w_n are statistical weights, the $p(k|n)$ are probability distributions.] Then

$$p(g, k) = \sum_n w_n p(g|n) p(k|n) \tag{17}$$

is *the right canonical decomposition* of $p(g, k)$. It has the following properties:

(i) The $p(k|n)$ are *disjoint*, i.e., $n \neq n' \Rightarrow \forall k \in K: p(k|n)p(k|n') = 0$.
(ii) The $p(g|n)$ are *distinct*, i.e., $\forall n \neq n'$: $\exists g \in G$: $p(g|n) \neq p(g|n')$.
(iii) The terms $p(g|n)p(k|n)$ that are mixed in (17) are *separable* (factorizable) describing statistical independence of G and K.

It is noteworthy that (17) is the *unique* decomposition of $p(g, k)$ satisfying (i)-(iii). This can be proved in a straightforward way by writing down another such decomposition, and by inferring, step-by-step, that the entities in it coincide with the corresponding entities in (17).

In a *strict* sense *separability* is synonymous with statistical independence (cf. Clauser and Horne, 1974). If $p(g, k)$ describes an ensemble of composite events (g, k), it is only the subensembles $p(g|n)p(k|n)$ that are thus separable. In the latter, the probability of an arbitrary event $P(g)$ (the characteristic function of a subset P of G) is $\sum_{g \in P} p(g|n)$ before the measurement of an arbitrary event $Q(k)$ (the characteristic function of a subset Q of K), and it is *unchanged* after the measurement [in the subensemble corresponding to the result $Q(k) = 1$].

In an arbitrary $p(g, k)$, however, the corresponding probabilities are $\sum_{g \in P} \sum_{k \in K} p(g, k)$ before the measurement and $\sum_{g \in P} \sum_{k \in Q} p(g, k)/p(Q)$ after the measurement. They need not be equal.

Nevertheless, one can still call the correlations in $p(g, k)$ *separable* in view of the fact that *the ignorance interpretation of the mixture* (17) is tenable. In other words, it is a subjective lack of knowledge that fails to place a particular composite event into one [and only one, due to (i)] of the subensembles enumerated by n in (17). In reality, one can argue, the composite event does belong to one of the subensembles, and hence the true probability of $P(g)$ is actually independent of whatever one does on subsystem II.

This was the $C + C$ (both discrete) case. Now we show that the entire argument is valid *mutatis mutandis* for the $Q + C$ case in the hybrid-state formalism.

Let (ρ_k, p_k) be a macrostate of $Q+C$, $\bar{K}$ the support of p_k, $\bar{K}_n \equiv \{k: p_k > 0,\ \rho_k \equiv \rho_n$ indep. of $k\}$, $n = 1, 2, \ldots$, giving $\sum_n \bar{K}_n = \bar{K}$. Finally, let

$$\forall n: \qquad w_n \equiv \sum_{k \in \bar{K}_n} p_k = p(\bar{K}_n)$$

and

$$\forall n,\ \forall k \in K: \qquad p(k|n) = \delta(k \in \bar{K}_n) p_k / w_n$$

Then

$$(\rho_k, p_k) = \sum_n w_n(\rho_n, p(k|n)) \tag{18}$$

[Note that the conditional states on the rhs of (18) are 0 for $k \notin \bar{K}_n$, and all equal to ρ_n otherwise.] Decomposition (18) can easily be checked with the help of (15a)-(15c).

Property (iii) of (18) consists in the fact that there is only one nonzero conditional state in each term on the rhs (instead of distinct ones for distinct k values).

Analogously as in the $C+C$ case, one proves that (18) is the only decomposition of (ρ_k, p_k) satisfying (i)-(iii).

The discussion of separability is now literally transferable from the $C+C$ to the $Q+C$ case.

Finally, in the $Q+Q$ case, in contrast, one has in general *nonseparability*. Actually, only if $\rho = \rho_{\mathrm{I}} \otimes \rho_{\mathrm{II}}$ does one have statistical independence of subsystems I and II and hence separability. If $\rho \in \sum_n w_n \rho_{\mathrm{I}}^{(n)} \otimes \rho_{\mathrm{II}}^{(n)}$ with at least two distinct nonzero terms, or is more intricate than that, the concept of separability in the broader sense is not applicable consistently, because the ignorance interpretation of the mixture fails to be consistent (cf. Fraassen, 1972).

7. CONCLUDING REMARKS

The very fact that one can perform both the σ-convex combinations (or the inverse decompositions) and the evaluation of expectation values (comprising also the probabilities of events) in terms of the correlation entities of macrostates (the hybrid states) of the $Q+C$ system, as proved in this work, should help to shed new light on the nature of quantum nonseparability. The importance of the latter has been established both theoretically and experimentally in distant correlation theory (cf. Clauser and Shimony, 1978; Aspect et al., 1981, 1982a,b).

Further, the hybrid, i.e., half quantum and half classical discrete, character of the new form (ρ_k, p_k) of the macrostates can, it is hoped, help to set up a bridge between the quantum and the classical descriptions.

(There is, of course, the additional step from the CD case to the CC—the classical continuous—one.)

As to applications of a Jauch-type hybrid macrostate formalism such as the one developed in this article, there are three obvious domains:

(i) The quantum theory of measurement (Bohr repeatedly emphasized that classical apparatuses are indispensable in measurement).
(ii) The quantum theory of a preparator, i.e., of a classical arrangement that, interacting with a certain quantum system, brings about a quantum state of the latter.
(iii) The structure of some quantum systems, such as molecules (which depend on the environment, with the latter, in turn, described most practically by classical physics; cf. Primas, 1983).

REFERENCES

Aspect, A., Grangier, P., and Roger, G. (1981). *Physical Review Letters*, **47**, 460.

Aspect, A., Grangier, P., and Roger, G. (1982a). *Physical Review Letters*, **49**, 91.

Aspect, A., Dalibard, J., and Roger, G. (1982b). *Physical Review Letters*, **49**, 1804.

Clauser, J. F., and Horne, M. A. (1974). *Physical Review D*, **10**, 526.

Clauser, J. F., and Shimony, A. (1978). *Reports on Progress in Physics*, **41**, 1881.

Fraassen, B. C. (1972). A formal approach to the philosophy of science, in *Paradigms and Paradoxes*, R. C. Colodny, ed., University of Pittsburgh Press.

Furry, W. H. (1936). *Physical Review*, **49**, 393.

Herbut, F. (1986). Critical investigation of Jauch's approach to the quantum theory of measurement, *International Journal of Theoretical Physics*, **25**, 863.

Herbut, F., and Vujičić, M. (1976). *Annals of Physics*, **96**, 382.

Jauch, J. M. (1964). *Helvetica Physica Acta*, **37**, 293.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, London.

Ozawa, M. (1984). *Journal of Mathematical Physics*, **25**, 79.

Primas, H. (1983). Quantum mechanics and chemistry, in *Les Fondements de la Mécanique Quantique*, Montana.

Von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, New Jersey.

Vujičić, M., and Herbut, F. (1984). *Journal of Mathematical Physics*, **25**, 2253.